MA 2733 Practice Exam 2 Solutions

1.) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}x^n$, by Ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)5^{n+1}}\cdot\frac{n5^n}{x^n}\right| = |x|\lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{5^n}{5^{n+1}}$

$= \frac{|x|}{5}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x|}{5}$. Need $\frac{|x|}{5}<1 \Rightarrow |x|<5$ so rad. of conv. $=5$. Check $x=5$: $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ which converges by AST, $x=-5$: $-\sum_{n=1}^{\infty}\frac{1}{n}$ divergent harmonic so int. of conv. $(-5,5]$.

2.) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{8^n}(x+6)^n$, by ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+6)^{n+1}\sqrt{n+1}}{8^{n+1}}\cdot\frac{8^n}{\sqrt{n}}\right| = |x+6|\lim_{n\to\infty}\frac{8^n}{8^{n+1}}\cdot\sqrt{\frac{n+1}{n}} = \frac{|x+6|}{8}\lim_{n\to\infty}\sqrt{\frac{n+1}{n}}$

$= \frac{|x+6|}{8}$. Need $\frac{|x+6|}{8}<1 \Rightarrow |x+6|<8$ so $R=8$. Then $-8<x+6<8 \Rightarrow -14<x<2$. $x=2$: $\sum_{n=1}^{\infty}\sqrt{n}$ and $\sqrt{n}\to\infty$ so divergent by test for divergence, $x=-14$: $\sum_{n=1}^{\infty}(-1)^n\sqrt{n}$ divergent by test for divergence so int. of conv. $(-14,2)$

3.) $\int \tan^{-1}x\,dx$. 1st $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \frac{1}{1--x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} \Rightarrow \tan^{-1}x = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx$

$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}+C$ but $\tan^{-1}(0)=0 \Rightarrow C=0$ so $\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \Rightarrow \int\tan^{-1}x\,dx = \int\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\,dx$

$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{2n+2}+C$

4.) $\int\frac{dx}{1+x^{11}} = \int\frac{1}{1--x^{11}}\,dx = \int\sum_{n=0}^{\infty}(-x^{11})^n\,dx = \int\sum_{n=0}^{\infty}(-1)^n x^{11n}\,dx = \sum_{n=0}^{\infty}(-1)^n\frac{x^{11n+1}}{11n+1}+C$

5.) $f(x)=\sqrt{x}$, $a=16$, then $f'(x)=\frac{1}{2\sqrt{x}}=\frac{1}{2}x^{-\frac{1}{2}}$, $f''(x)=-\frac{1}{2^2}x^{-\frac{3}{2}}$, $f'''(x)=\frac{3}{2^3}x^{-\frac{5}{2}}$

$\cdots f^{(n)}(x)=\frac{(-1)^{n-1}}{2^n}(1\cdot3\cdots(2n+1))x^{-\frac{2n-1}{2}}$ for $n\geq1$ so $f^{(n)}(a)=\frac{(-1)^{n-1}1\cdot3\cdots(2n+1)}{2^n4^{2n-1}} = (-1)^{n-1}\frac{1\cdot3\cdots(2n+1)}{2^{5n-2}}$ $n\geq1$

so $f(x)=4+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1\cdot3\cdots(2n+1)}{2^{5n-2}n!}(x-16)^n$

6.) $f(x)=e^{-2x}$ at $a=1$. $f'(x)=-2e^{-2x}$, $f''(x)=2^2e^{-2x}$, $f'''(x)=-2^3e^{-2x}$, $\ldots$ $f^{(n)}(x)=(-1)^n2^ne^{-2x}$

so $f^{(n)}(a)=(-1)^n2^ne^{-2}$ so $f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{2^ne^{-2}}{n!}(x-1)^n$

7.) $x=t^3-3t$, $y=t^3-3$, then $\frac{dx}{dt}=3t^2-3=3(t-1)(t+1)$. $\frac{dy}{dt}=3t^2$

Then horiz. tang: $\frac{dx}{dt}\neq0$, $\frac{dy}{dt}=0 \Rightarrow t=0$ and $t\neq1,-1$ so pt of the curve $(0,-3)$

vert. tang: $\frac{dx}{dt}=0$, $\frac{dy}{dt}\neq0 \Rightarrow t\neq0$, $t=-1,1$ so pts $(2,-4)$, $(-2,-2)$

8.) $x=\cos\theta$, $y=\cos(3\theta)$, so $\frac{dx}{d\theta}=-\sin\theta$, $\frac{dy}{d\theta}=-3\sin(3\theta)$. $\begin{cases}\sin\theta=0 \Rightarrow \theta=\pi n \text{ for } n\in\mathbb{Z}\\ \sin(3\theta)=0 \Rightarrow \theta=\frac{n\pi}{3} \text{ for } n\in\mathbb{Z}\end{cases}$

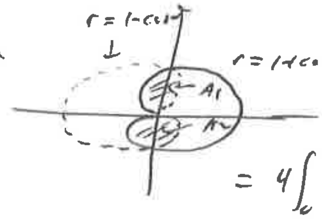
so horiz tang: $\frac{dx}{d\theta}\neq0$, $\frac{dy}{d\theta}=0 \Rightarrow \theta=\frac{n\pi}{3}$, $\theta\neq n\pi$, issue at $\theta=0$, $\frac{0}{0}$ type

so $\lim_{\theta\to0}\frac{-3\sin(3\theta)}{-\sin\theta} = \lim_{\theta\to0}\frac{9\cos(3\theta)}{\cos\theta} = 9$ by L'hôpital. $\theta=0$ neither hor. or ver.

so pts are $(\cos(\frac{n\pi}{3}), \cos(n\pi))$

vert tang: $\theta\neq\frac{n\pi}{3}$, $\theta=n\pi$ so pts are $(\cos(n\pi), \cos(3n\pi))$

9.) $r = 1+\cos\theta$, $r = 1-\cos\theta$ then

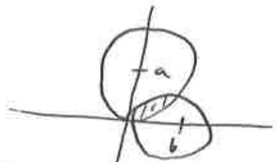


$r = 1+\cos\theta$ then $A_1 = A_2$ and $A = A_1 + A_2 = 2A_2$

so $A_1 = \int_0^{\frac{\pi}{2}} \left((1+\cos\theta)^2 - (1-\cos\theta)^2 \right) d\theta$

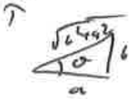
$= 4\int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = 4\sin\theta \Big|_0^{\frac{\pi}{2}} = 4$ so $A = 8$

10.) $r = a\sin\theta$, $r = b\cos\theta$, $a, b > 0$ then



$\Rightarrow a\sin\theta = b\cos\theta \Rightarrow \frac{b}{a} = \tan\theta$

$\Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$ in QI



so $A = \int_0^{\tan^{-1}(\frac{b}{a})} \left(b^2\cos^2\theta - a^2\sin^2\theta \right) d\theta$

$= \int_0^{\tan^{-1}(\frac{b}{a})} \left(b^2\cos^2\theta - a^2(1-\cos^2\theta) \right) d\theta = \int_0^{\tan^{-1}(\frac{b}{a})} \left((a^2+b^2)\cos^2\theta - a^2 \right) d\theta = \int_0^{\tan^{-1}(\frac{b}{a})} \left(\frac{a^2+b^2}{2}(1+\cos(2\theta)) - a^2 \right) d\theta$

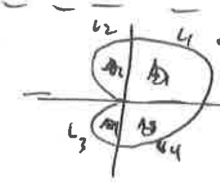
$= \frac{a^2+b^2}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_0^{\tan^{-1}(\frac{b}{a})} - a^2\theta \Big|_0^{\tan^{-1}(\frac{b}{a})} = \frac{b^2-a^2}{2}\tan^{-1}\left(\frac{b}{a}\right) + \frac{1}{4}\sin\left(2\tan^{-1}\left(\frac{b}{a}\right)\right) = \frac{b^2-a^2}{2}\tan^{-1}\left(\frac{b}{a}\right) + \frac{1}{2}\sin\left(\tan^{-1}\left(\frac{b}{a}\right)\right)\cos\left(\tan^{-1}\left(\frac{b}{a}\right)\right)$

$= \frac{b^2-a^2}{2}\tan^{-1}\left(\frac{b}{a}\right) + \frac{1}{2}\frac{b}{\sqrt{a^2+b^2}}\frac{a}{\sqrt{a^2+b^2}}$

11.) $r = 2\sin\theta$, $0 \le \theta \le 2\pi$, then $r'(\theta) = 2\cos\theta$ so $r^2 + (r')^2 = 4\sin^2\theta + 4\cos^2\theta = 4$

$\therefore L = \int_0^{2\pi} \sqrt{r^2+(r')^2}\, d\theta = 2\int_0^{2\pi} d\theta = 4\pi$

12.) $r = 2(1+\cos\theta)$ then



$r = 2(1+\cos\theta)$ then $L = L_1 + L_2 + L_3 + L_4$ ~~$A = A_1 + A_2 + A_3 + A_4$~~ and ~~$A_1 = A_3$, $A_2 = A_4$~~ $L_1 = L_4$, $L_2 = L_3$

so $r' = -2\sin\theta$ $r^2 + (r')^2 = 4(1 + 2\cos\theta + \cos^2\theta) + 4\sin^2\theta$

$= 4(2+2\cos\theta) = 8(1+\cos\theta) = 16\cos^2\left(\frac{\theta}{2}\right)$

so $L_1 = \int_0^{\frac{\pi}{2}} \sqrt{16\cos^2\left(\frac{\theta}{2}\right)}\, d\theta = 4\int_0^{\frac{\pi}{2}} \cos\left(\frac{\theta}{2}\right) d\theta = 8\sin\left(\frac{\theta}{2}\right)\Big|_0^{\frac{\pi}{2}} = 8\frac{\sqrt{2}}{2} = 4\sqrt{2}$

$L_2 = \int_{\frac{\pi}{2}}^{\pi} \sqrt{16\cos^2\left(\frac{\theta}{2}\right)}\, d\theta = 4\int_{\frac{\pi}{2}}^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 8\sin\left(\frac{\theta}{2}\right)\Big|_{\frac{\pi}{2}}^{\pi} = 8\left(\frac{2-\sqrt{2}}{2}\right) = 4(2-\sqrt{2})$

so $L = 8\sqrt{2} + 8(2-\sqrt{2})$
$= 16$

---

13) have $\sum_{n=0}^{\infty} c_n x^n$ so by ~~ratio~~ root test $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|c_n||x|^n} = |x| \lim_{n\to\infty} \sqrt[n]{|c_n|} = |x|c$

need $|x|c < 1 \Rightarrow |x| < \frac{1}{c} \Rightarrow R = \frac{1}{c}$